Singularities, Lax degeneracies and Maslov indices of the periodic Toda chain

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Abstract

The n-particle periodic Toda chain is a well known example of an integrable but nonseparable Hamiltonian system in $\mathbb{R}^{2n}$. We show that Σ_k, the k-fold singularities of the Toda chain, ie points where there exist k independent linear relations amongst the gradients of the integrals of motion, coincide with points where there are k (doubly) degenerate eigenvalues of representatives L and $\overline{\mathrm{L}}$ of the two inequivalent classes of Lax matrices (corresponding to degenerate periodic or antiperiodic solutions of the associated second-order difference equation). The singularities are shown to be nondegenerate, so that Σ_k is a codimension-$2k$ symplectic submanifold. Σ_k is shown to be of elliptic type, and the frequencies of transverse oscillations under Hamiltonians which fix Σ_k are computed in terms of spectral data of the Lax matrices.

If $\mu(C)$ is the (even) Maslov index of a closed curve C in the regular component of $\mathbb{R}^{2n}$, then $(-1)^{\mu(C)/2}$ is given by the product of the holonomies (equal to ± 1) of the even- (or odd-) indexed eigenvector bundles of L and $\overline{\mathrm{L}}$.

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1 Introduction

The Toda chain is a canonical example of a nonseparable but integrable Hamiltonian system. It consists of n particles on the line with exponential interactions between consecutively labeled particles. In the periodic Toda chain, the nth particle is coupled to the first. The Hamiltonian is

$$H = \sum_{j=1}^{n} \tfrac{1}{2}(p_j^2 + b_j^2), \tag{1.1}$$

where

$$b_j = e^{(q_j - q_{j+1})/2}, \tag{1.2}$$

and $q_{n+1} \equiv q_1$.

The Toda lattice was introduced in [24]. Its integrability was established by Hénon [19] and Flaschka [13] using the method of Lax pairs. There now exists an extensive literature on the problem (see, eg, [14]). Properties of eigenvectors of Lax matrices over their associated spectral curve were studied in classical papers by Adler and van Moerbeke [1, 2, 3] and van Moerbeke and Mumford [26]. Audin [6] has used these results to analyse the topology of the set of regular values of the integrals of motion. A recent account is given by Babelon, Bernard and Talon [7].

In the last 20 years there has been much interest in the topology of integrable finite-dimensional Hamiltonian systems. The generic local structure and dynamics is given by the Liouville-Arnold theorem [5], according to which neighbourhoods of phase space are foliated into invariant Lagrangian submanifolds diffeomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$, where $\mathbb{T}^k$ is the k-torus, and the dynamics is linearised by action-angle coordinates. This local behaviour breaks down at critical points of the energy-momentum map, which comprise invariant sets of lower dimension. A Morse theory for integrable Hamiltonian systems, wherein the global topology is described in terms of these critical sets, has been extensively developed by Fomenko [15], Eliasson [12], Vey [27] and Tien Zung [28], among others.

Our interest here is in the phase space topology of the Toda chain. The paper is organised as follows. In Section 2.1, we obtain Lax generators for the flows generated by a complete set of n integrals $F : \mathbb{R}^{2n} \to \mathbb{R}^n$. These Lax generators are constructed from representatives L and $\overline{\mathrm{L}}$ of the two inequivalent classes of symmetric Lax matrices for the periodic Toda chain (these classes correspond to periodic and antiperiodic solutions of the associated second-order difference equation). Our formulation is related to the classical treatment of van Moerbeke and Mumford of general periodic finite difference operators [26], but our approach is self-contained and quite elementary.

These results are used to establish, in Section 3, a one-to-one correspondence between singularities of the Toda flow (critical points of F) and eigenvalue degeneracies of the Lax matrices L and $\overline{\mathrm{L}}$. More precisely, the corank of dF is equal to the number of (doubly) degenerate eigenvalues of L and $\overline{\mathrm{L}}$. Similar ideas relating singularities to points of degeneracy on the spectral curve are discussed by Audin [6].

In Section 4 we determine the local structure of the corank-k singularities Σ_k. These are shown to be codimension-$2k$ symplectic submanifolds of elliptic type composed of $\binom{n-1}{k}$ components disconnected from each other. The eigenvalues of the linearised integrable flows which fix them are computed in terms of the spectral data of L and $\overline{\mathrm{L}}$. There are singularities of corank between 1 and $(n-1)$, and Σ_k is contained in the closure of Σ_j for $k > j$. For $n = 3$, related results on the singularities and critical values for the Toda chain and its algebraic generalisations have been obtained by Polyakova [22] following the programme of Fomenko [15].

The Maslov index is a topological invariant of Lagrangian tori of integrable systems which appears in the semiclassical (EBK) quantisation conditions [21, 20]. In a companion paper [16], we show that the Maslov index is determined by corank-one singularities. In Section 5, it is shown that the (even) Maslov index of a closed curve in the set of regular points of F is determined, modulo 4, by the product of the even- (or odd-) indexed holonomies of the eigenvector bundles of L and $\overline{\mathrm{L}}$. We note that, because the Toda chain is nonseparable, application of the semiclassical quantisation rules is not straightforward.

2 Higher Lax flows

The Lax formulation of the equations of motion for the Toda chain was obtained by Flaschka [13]. Let L be the $n \times n$ symmetric matrix given by

$$\mathrm{L} = \begin{pmatrix} p_1 & b_1 & 0 & \cdots & 0 & b_n \\ b_1 & p_2 & b_2 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & b_{n-2} & p_{n-1} & b_{n-1} \\ b_n & 0 & \cdots & 0 & b_{n-1} & p_n \end{pmatrix}, \tag{2.1}$$

and $\mathrm{M}_{(2)}$ the $n \times n$ antisymmetric matrix given by

$$\mathrm{M}_{(2)} = \frac{1}{2} \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 & -b_n \\ -b_1 & 0 & b_2 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -b_{n-2} & 0 & b_{n-1} \\ b_n & 0 & \cdots & 0 & -b_{n-1} & 0 \end{pmatrix}. \tag{2.2}$$

(To simplify notion, in particular in Section 4, we will sometimes write M instead of $\mathrm{M}_{(2)}$.) It is straightforward to verify that Hamilton's equations for the Hamiltonian (1.1) imply the Lax equation

$$\dot{\mathrm{L}} := \{\mathrm{L}, H\} = [\mathrm{L}, \mathrm{M}_{(2)}]. \tag{2.3}$$

Conversely, (2.3) along with the (independent) equation $\sum_{r=1}^n p_r = \sum_{r=1}^n \dot{q}_r$ imply Hamilton's equations. Thus, Hamilton's equations and the Lax equation are essentially equivalent. It is convenient to express L and $\mathrm{M}_{(2)}$ in index form,

$$L_{rs} = p_r \delta_{rs} + b_r \delta_{r+1,s} + b_s \delta_{r,s+1}, \tag{2.4a}$$
$$M_{(2)rs} = b_r \delta_{r+1,s} - b_s \delta_{r,s+1}. \tag{2.4b}$$

For convenience, here and elsewhere, matrices and vectors are regarded as being periodic in their indices, with period n. Thus $L_{r+n,s} = L_{r,s+n} = L_{rs}$. Similarly, we take δ_{rs} to be one if $r = s \mod n$ and to be zero otherwise.

As is well known, the Lax equation implies that the eigenvalues of L, as well as functions of them, are constants of the motion. In particular, the n functions

$$F_j = \tfrac{1}{j} \operatorname{Tr} \mathrm{L}^j, \quad 1 \le j \le n, \tag{2.5}$$

are conserved. We note that

$$F_1 = p_1 + \cdots + p_n \tag{2.6}$$

is the centre-of-mass momentum, while F_2 is the Hamiltonian.

In this section we construct Lax equations for the Hamiltonian flows generated by each of the F_j's. That is, we find antisymmetric matrices $\mathrm{M}_{(j)}$ such that

$$\{\mathrm{L}, F_j\} = [\mathrm{L}, \mathrm{M}_{(j)}] \tag{2.7}$$

We may take $\mathrm{M}_{(1)} = 0$ (since F_1 generates uniform translations and L is translation-invariant), while $\mathrm{M}_{(2)}$ is given by (2.1). The formulation presented here is related to that of van Moerbeke and Mumford [26], who give

nilpotent (triangular) Lax generators for the higher flows of general periodic finite difference operators. Before giving expressions for $\mathrm{M}_{(j)}$ for $j > 2$, we recall that the higher Lax equations (2.7) already imply that the functions F_j are in involution. Indeed,

$$\{F_{j+1}, F_k\} = \mathrm{Tr}\,\left(\mathrm{L}^j\{\mathrm{L}, F_k\}\right) = \mathrm{Tr}\,\left(\mathrm{L}^j[\mathrm{L}, \mathrm{M}_{(k)}]\right) = 0, \tag{2.8}$$

where the last equality follows from the cyclicity of the trace. We recall, too, that integrability does not follow immediately from (2.8); one also needs to show that the F_j's are functionally independent (here, functional independence is implied by Theorem 4.1 below).

We note that H is invariant under the n substitutions

$$b_j \mapsto -b_j. \tag{2.9}$$

It follows that the Lax equation (2.3) is similarly invariant under (2.9). It is easily seen that an even number of such substitutions can be generated by conjugations $\mathrm{L} \mapsto \mathrm{SLS}^{-1}$, $\mathrm{M}_{(2)} \mapsto \mathrm{SM}_{(2)}\mathrm{S}^{-1}$ for S a diagonal matrix of ± 1's. The Lax equation (2.3) is trivially invariant under such conjugations, as are the F_j's. There are, therefore, two inequivalent classes of Lax pairs with respect to (2.9). These are characterised by whether the number of negative b_j's is even or odd. A spectral characterisation of these even and odd classes is provided by the difference equation

$$b_r v_{r+1} + p_r v_r + b_{r-1} v_{r-1} = \lambda v_r. \tag{2.10}$$

Eigenvalues λ (with eigenvectors $\mathbf{v}$) of even L's correspond to periodic solutions $v_{r+n} = v_r$ of (2.10), whereas eigenvalues of odd L's correspond to antiperiodic solutions $v_{r+n} = -v_r$. For definiteness and convenience, we take L, as given in (2.1), to be the even representative, and $\overline{\mathrm{L}}$ to be given by replacing b_n by $-b_n$, ie

$$\overline{L}_{rs} = p_r \delta_{rs} + \sigma_r b_r \delta_{r+1,s} + \sigma_s b_s \delta_{r,s+1}, , \tag{2.11}$$

where

$$\sigma_r = \begin{cases} -1, & r = 0 \mod n, \\ 1, & \text{otherwise}, \end{cases} \tag{2.12}$$

to be the odd representative.

We say that an $n \times n$ symmetric matrix A is *off-banded of width* j, or j*-off-banded*, if it has precisely $(n-j)$ consecutive zero diagonals on or above

main diagonal (thus, j is the number of diagonals above these zero diagonals, the first of which does not vanish). Equivalently, A is j-off-banded if

$$A_{r,r+d} = 0, \quad 1 \le r \le n, \quad 0 \le d < \min(n-j-1, n-r), \tag{2.13a}$$

$$\sum_{r=1}^{j} |A_{r,r+n-j}| \ne 0. \tag{2.13b}$$

Proposition 2.1. *For $1 \le j \le n$, $\mathrm{L}^j - \overline{\mathrm{L}}^j$ is j-off-banded. Moreover, its elements on the first nonzero diagonal are given by*

$$\left(\mathrm{L}^j - \overline{\mathrm{L}}^j\right)_{r,r+j} = 2b_{r-1}b_{r-2}\cdots b_{r-j}, \qquad j < n, \tag{2.14a}$$

$$= 4, \qquad j = n, \tag{2.14b}$$

where $1 \le r \le j$.

Proof. Fix $1 \le r \le n$, and take d such that $0 \le d \le \min(n-j-1, n-r)$. From (2.11) and (2.12) it is clear that $(\mathrm{L}^j - \overline{\mathrm{L}}^j)_{r,r+d}$ is given by twice the sum of terms of $(\mathrm{L}^j)_{r,r+d}$ which are of odd degree in b_n. The terms of $(\mathrm{L}^j)_{r,r+d}$ are products $T(t_0, \ldots, t_j)$ of the form

$$T(t_0, \ldots, t_j) = L_{t_0 t_1} L_{t_1 t_2} \cdots L_{t_{j-1} t_j}, \tag{2.15}$$

where $t_0 = r$ and $t_j = r + d$. L_{1n} and L_{n1}, if they appear, contribute the only factors of b_n to $T(t_0, \ldots, t_j)$.

Since L is banded modulo n (cf (2.4a)), $T(t_0, \ldots, t_j)$ vanishes unless each pair of consecutive indices t_k and t_{k+1} in (2.15) differ by 0 or ± 1 modulo n. Let us call the factor $L_{t_k t_{k+1}}$ a *right step* if $t_{k+1} = t_k + 1 \mod n$, and a *left step* if $t_{k+1} = t_k - 1 \mod n$ (diagonal factors, for which $t_{k+1} = t_k \mod n$, are neither left nor right steps). Let u denote the number of right steps minus the number of left steps in $T(t_0, \ldots, t_j)$. Then

$$u = d \mod n. \tag{2.16}$$

Since $-j \le u \le j$ and, by assumption, $1 \le j \le n$ and $0 \le d < n$, it follows that either $u = d$ or $u = d - n$.

First, suppose that $u = d - n$. We show that $T(t_0, \ldots, t_j)$ is of odd degree in b_n. This is certainly true for terms which contain no right steps. Such terms are products of diagonal elements of L (which do not contribute factors of b_n) and the left-step-only product

$$L_{r,r-1} L_{r-1,r-2} \cdots L_{r+(n-d)+1, r+(n-d)}. \tag{2.17}$$

b_n appears just once in (2.17), in the factor $L_{10} := L_{1n}$. A general term $T(t_0, \ldots, t_j)$ with $u = d - n$ is a product of diagonal elements, the left-step-only product (2.17), and palindromic products of off-diagonal elements, ie, products of the form

$$L_{k,k+1} L_{k+1,k+2} \cdots L_{k+p-1,k+p} \cdot L_{k+p,k+p-1} \cdots L_{k+1,k+2} L_{k,k+1} \tag{2.18}$$

in which every factor appears twice.

Next, suppose that $d = u$. A similar argument implies that $T(t_0, \ldots, t_j)$ is of even degree in b_n. In this case, we note that the right-step-only product $L_{r,r+1} L_{r+1,r+2} \cdots L_{r+d-1,r+d}$ contains no factors of L_{1n} or L_{n1}, and that, in general, $T(t_0, \ldots, t_j)$ is a product of diagonal elements, the right-step-only product, and palindromic products (2.18).

Thus, $T(t_0, \ldots, t_j)$ is of odd degree in b_n if and only if $u = d - n$, or, equivalently, $d = n - |u|$. Since $|u| \leq j$, this condition can be satisfied only if $d \geq n - j$. Thus, if $d < n - j$, then $(\mathrm{L}^j - \overline{\mathrm{L}}^j)_{r,r+d} = 0$, in accord with (2.13a).

To establish (2.13b), we verify (2.14), which implies that $\mathrm{L}^j - \overline{\mathrm{L}}^j$ has nonzero elements on the $(n-j)$th diagonal above the main diagonal. Let $d = n - j$. For $T(t_0, \ldots, t_j)$ to be of odd degree in b_n, we require that $u = j$. For $j < n$, there is only one such term, namely the left-step-only product $L_{r,r-1} L_{r-1,r-2} \cdots L_{r-j+1,r-j} = b_{r-1} b_{r-2} \cdots b_{r-j}$. (2.14a) follows. For $j = n$, there are two nonzero terms with $u = n$. The first is the left-step-only product $L_{r,r-1} L_{r-1,r-2} \cdots L_{r-n+1,r-n} = b_{r-1} b_{r-2} \cdots b_{r-n} = 1$. The second is the right-step-only product $L_{r,r+1} L_{r+1,r+2} \cdots L_{r+n-1,r+n} = b_{r+1} b_{r+2} \cdots b_r = 1$. (2.14b) follows. $\square$

From Proposition 2.1 it follows immediately that

$$\operatorname{Tr} \mathrm{L}^j = \begin{cases} \operatorname{Tr} \overline{\mathrm{L}}^j, & 1 \leq j < n, \\ \operatorname{Tr} \overline{\mathrm{L}}^n + 4n, & j = n. \end{cases} \tag{2.19}$$

Also, we note that if a linear combination of off-banded matrices vanishes and the matrices all have different widths, then the coefficient of each matrix vanishes (argue inductively, starting with the matrix of greatest width). Therefore, Proposition 2.1 also implies the following:

Corollary 2.1. *If* $\sum_{j=2}^n c_j (\mathrm{L}^{j-1} - \overline{\mathrm{L}}^{j-1}) = 0$, *then* $c_2 = \cdots = c_n = 0$.

We introduce the following notation: given an $n \times n$ matrix A, let A_+ denote its strictly upper triangular part, ie the matrix elements $A_{r,r+d}$ with $1 \leq r \leq n$ and $1 \leq d \leq n - r$. The generators of the Lax flows for the F_j's are given by the following:

Proposition 2.2. *Let* $\mathrm{M}_{(j)}$ *and* $\overline{\mathrm{M}}_{(j)}$ *be the antisymmetric matrices given by*

$$\mathrm{M}_{(j)+} = \tfrac{1}{2}(\overline{\mathrm{L}}^{j-1})_+, \tag{2.20a}$$
$$\overline{\mathrm{M}}_{(j)+} = \tfrac{1}{2}(\mathrm{L}^{j-1})_+ \tag{2.20b}$$

for $1 \le j \le n$*. Then*

$$\{\mathrm{L}, F_j\} = [\mathrm{L}, \mathrm{M}_{(j)}], \tag{2.21a}$$
$$\{\overline{\mathrm{L}}, F_j\} = [\overline{\mathrm{L}}, \overline{\mathrm{M}}_{(j)}]. \tag{2.21b}$$

We note that $\mathrm{M}_{(1)}$ and $\overline{\mathrm{M}}_{(1)}$ both vanish (consistent with the fact that L is invariant under uniform translations) while for $j = 2$, (2.20a) agrees with (2.1).

Proof. First, we note that (2.19) implies that under the substitution $b_n \mapsto -b_n$, F_j changes by at most a constant. Therefore, the two sets of Lax equations equations (2.21a) and (2.21b) are related by this substitution, and it suffices to verify just one of them. For definiteness we consider (2.21b).

Since both sides are symmetric matrices, it suffices to verify (2.21b) for elements on or above the main diagonal. The elements of the left-hand side are given by

$$\{\overline{L}_{rs}, F_j\} = \frac{1}{j}\mathrm{Tr}\,\{\overline{L}_{rs}, \mathrm{L}^j\} = \mathrm{Tr}\,\left(\{\overline{L}_{rs}, \mathrm{L}\}\mathrm{L}^{j-1}\right). \tag{2.22}$$

From (2.4a), (2.11) and

$$\{b_r, p_s\} = \tfrac{1}{2}b_r(\delta_{rs} - \delta_{r+1,s}), \tag{2.23}$$

a straightforward calculation gives, for $1 \le r \le s \le n$, that

$$\{\overline{L}_{rs}, F_j\} = \begin{cases} b_{r-1}(\mathrm{L}^{j-1})_{r-1,r} - b_r(\mathrm{L}^{j-1})_{r,r+1}, & r = s, \\ \frac{1}{2}b_r\left((\mathrm{L}^{j-1})_{rr} - (\mathrm{L}^{j-1})_{r+1,r+1}\right), & r+1 = s, \\ \frac{1}{2}b_n\left((\mathrm{L}^{j-1})_{11} - (\mathrm{L}^{j-1})_{nn}\right), & r = 1, s = n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.24}$$

Next we evaluate the right-hand side of (2.21b), ie $[\overline{\mathrm{L}}, \overline{\mathrm{M}}_{(j)}]$. From (2.11),

$$[\overline{\mathrm{L}}, \overline{\mathrm{M}}_{(j)}]_{rs} = p_r\overline{M}_{(j)rs} + \sigma_{r-1}b_{r-1}\overline{M}_{(j)r-1,s} + \sigma_r b_r\overline{M}_{(j)r+1,s} + (r \leftrightarrow s). \tag{2.25}$$

For $r = s$, (2.25) yields $b_{r-1}\mathrm{L}^{j-1}{}_{r-1,r} - b_r\mathrm{L}^{j-1}{}_{r+1,r}$, in agreement with (2.24) (note that $\sigma_{r-1}\overline{M}_{(j)r-1,r} = \frac{1}{2}\mathrm{L}^{j-1}{}_{r-1,r}$, and similarly, $\sigma_r\overline{M}_{(j)r+1,r} = -\frac{1}{2}\mathrm{L}^{j-1}{}_{r+1,r}$).

To evaluate the off-diagonal elements $1 \le r < s \le n$ in (2.24), we will make use of the trivial identity $[\mathrm{L}, \frac{1}{2}\mathrm{L}^{j-1}] = 0$, or

$$[\mathrm{L}, \tfrac{1}{2}\mathrm{L}^{j-1}]_{rs} = \tfrac{1}{2}p_r(\mathrm{L}^{j-1})_{rs} + \tfrac{1}{2}b_{r-1}(\mathrm{L}^{j-1})_{r-1,s} + \tfrac{1}{2}b_r(\mathrm{L}^{j-1})_{r+1,s} - (r \leftrightarrow s) = 0. \tag{2.26}$$

Referring to the terms in (2.25), we use (2.20b) to express M_j in terms of L^{j-1}, as follows:

$$\begin{aligned} p_r\overline{M}_{(j)rs} + (r \leftrightarrow s) &= p_r(\tfrac{1}{2}\mathrm{L}^{j-1})_{rs} - (r \leftrightarrow s), \\ \sigma_{r-1}b_{r-1}\overline{M}_{(j)r-1,s} + (r \leftrightarrow s) &= b_{r-1}(\tfrac{1}{2}\mathrm{L}^{j-1})_{r-1,s} - (r \leftrightarrow s) \\ &\quad - \delta_{r1}\delta_{sn}b_n(\tfrac{1}{2}\mathrm{L}^{j-1})_{nn} + \delta_{r+1,s}b_r(\tfrac{1}{2}\mathrm{L}^{j-1})_{rr}, \\ \sigma_r b_r\overline{M}_{(j)r+1,s} + (r \leftrightarrow s) &= b_r(\tfrac{1}{2}\mathrm{L}^{j-1})_{r+1,s} - (r \leftrightarrow s) \\ &\quad + \delta_{r1}\delta_{sn}b_n(\tfrac{1}{2}\mathrm{L}^{j-1})_{11} - \delta_{r+1,s}b_r(\tfrac{1}{2}\mathrm{L}^{j-1})_{r+1,r+1}. \end{aligned} \tag{2.27}$$

Substituting the preceding into (2.25) and using the identity (2.26), we get that

$$\begin{aligned} [\overline{\mathrm{L}}, \overline{\mathrm{M}}_{(j)}]_{rs} = [\mathrm{L}, \tfrac{1}{2}\mathrm{L}^{j-1}]_{rs} + \tfrac{1}{2}\delta_{r+1,s}b_r((\mathrm{L}^{j-1})_{rr} - (\mathrm{L}^{j-1})_{r+1,r+1}) + \\ \tfrac{1}{2}b_n\delta_{r1}\delta_{sn}((\mathrm{L}^{j-1})_{11} - (\mathrm{L}^{j-1})_{nn}). \end{aligned} \tag{2.28}$$

As the first term vanishes, this agrees with (2.24). □

3 Singularities and eigenvalue degeneracies

The singularities of an integrable system $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ are the critical points of F. Here we show that singularities of the Toda chain coincide with eigenvalue degeneracies of the Lax matrices L and $\overline{\mathrm{L}}$. Let Σ denote the set of singularities of the Toda chain, and let $\Sigma_k \subset \Sigma$ denote the subset in which there are precisely k linear relations amongst the dF_j's, ie

$$\Sigma_k = \{(\mathbf{q}, \mathbf{p}) |\, \mathrm{corank}\, dF = k\}. \tag{3.1}$$

We observe that eigenvalues of L and $\overline{\mathrm{L}}$ are at most two-fold degenerate, since the associated eigenvectors are solutions of the second-order linear difference equation (2.10), which for a given value of λ has at most two linearly independent solutions. Let ν and $\overline{\nu}$ denote the number of doubly degenerate eigenvalues of L and $\overline{\mathrm{L}}$ respectively.

Theorem 3.1.

$$\mathrm{corank}\, dF = \nu + \overline{\nu}. \tag{3.2}$$

Proof. Let $\mathbf{z}_* \in \mathbb{R}^{2n}$. For convenience, let F_* denote $F(\mathbf{z}_*)$, L_* denote $\mathrm{L}(\mathbf{z}_*)$, etc. Let $V_* \subset \mathbb{R}^n$ denote the space of linear relations amongst the dF_{j*}'s, ie

$$V_* = \left\{ \mathbf{c} \in \mathbb{R}^n \middle| \sum_{j=1}^n c_j dF_{j*} = 0 \right\}. \tag{3.3}$$

so that $\operatorname{corank} dF_* = \dim V_*$. We show that

$$\dim V_* = \nu_* + \overline{\nu}_*. \tag{3.4}$$

We first show that $\dim V_* \le \nu_* + \overline{\nu}_*$. Let $\mathcal{A}_*$ denote the real antisymmetric commutant of L_*; that is, $\mathcal{A}_*$ consists of all n-dimensional real antisymmetric matrices which commute with L_*. For $\mathbf{c} \in V_*$, Proposition 2.2 and Eq. (3.3) imply that

$$\left[\mathrm{L}_*, \sum_{j=1}^n c_j \mathrm{M}_{(j)*} \right] = \left\{ \mathrm{L}, \sum_{j=1}^n c_j F_j \right\}_* = 0, \tag{3.5}$$

so that $\sum_{j=1}^n c_j \mathrm{M}_{(j)*} \in \mathcal{A}_*$. Similarly, letting $\overline{\mathcal{A}}_*$ denote the real antisymmetric commutant of $\overline{\mathrm{L}}$, we have that $\sum_{j=1}^n c_j \overline{\mathrm{M}}_{(j)*} \in \overline{\mathcal{A}}_*$. Regarding $\mathcal{A}_*$ and $\overline{\mathcal{A}}_*$ as real vector spaces, we define a linear map from V_* to $\mathcal{A}_* \oplus \overline{\mathcal{A}}_*$ according to

$$\mathbf{c} \mapsto \left(\sum_{j=1}^n c_j \mathrm{M}_{(j)*} \right) \oplus \left(\sum_{j=1}^n c_j \overline{\mathrm{M}}_{(j)*} \right). \tag{3.6}$$

This map is 1-1, for if $\sum_{j=1}^n c_j \mathrm{M}_{(j)*} = \sum_{j=1}^n c_j \overline{\mathrm{M}}_{(j)*} = 0$, then from (2.20) and Corollary 2.1 it follows that

$$c_2 = \cdots = c_n = 0. \tag{3.7}$$

But (3.7) and (3.3) imply that $c_1 dF_1 = 0$. Since $dF_1 \neq 0$ (cf (2.6)), we must have $c_1 = 0$, and therefore $\mathbf{c} = 0$. Thus, (3.6) is 1-1, and

$$\dim V_* \le \dim \mathcal{A}_* + \dim \overline{\mathcal{A}}_*. \tag{3.8}$$

To compute $\dim \mathcal{A}_*$, we note that $\mathcal{A}_*$ is the direct sum of the spaces of antisymmetric linear maps on the eigenspaces of L_* (endowed with the standard inner product from $\mathbb{R}^n$). In general, the space of antisymmetric linear maps on a k-dimensional inner product space is of dimension $k(k-1)/2$. Since L_* has ν_* two-dimensional eigenspaces and $(n - 2\nu_*)$ one-dimensional eigenspaces, it follows that $\dim \mathcal{A}_* = \nu_*$. Similarly, $\dim \overline{\mathcal{A}}_* = \overline{\nu}_*$. Substitution into (3.8) yields

$$\dim V_* \le \nu_* + \overline{\nu}_*. \tag{3.9}$$

Next, we show that $\nu_* + \overline{\nu}_* \leq \dim V_*$. Let $\mathcal{T}_*$ denote the set of polynomials which annihilate L_*. Elements of $\mathcal{T}_*$ are divisible by the minimum polynomial of L_*, which we denote by $P_*(x)$. $P_*(x)$ is of degree $n - \nu_*$. Regarding $\mathcal{T}_*$ as a vector space, we let $\mathcal{T}_*^n$ denote the subspace of polynomials of degree at most n, ie

$$\mathcal{T}_*^n = \{R(x) | R(\mathrm{L}_*) = 0, \deg R \leq n\}. \tag{3.10}$$

Clearly $\dim \mathcal{T}_*^n = \nu_*$ (elements of $\mathcal{T}_*^n$ are products of $P_*(x)$ with arbitrary polynomials of degree at most ν_*). Suppose $\sum_{j=1}^n c_j x^{j-1} \in \mathcal{T}_{n*}$. Then

$$\sum_{j=1}^n c_j {\mathrm{L}_*}^{j-1} = 0 \implies \mathrm{Tr}\,(\sum_{j=1}^n c_j {\mathrm{L}_*}^{j-1} d\mathrm{L}_*) = 0 \implies \sum_{j=1}^n c_j dF_{j*} = 0. \tag{3.11}$$

Therefore,

$$\sum_{j=1}^n c_j x^{j-1} \mapsto \mathbf{c} \tag{3.12}$$

defines a linear map from $\mathcal{T}_*^n$ to V_*. Clearly the map (3.12) is 1-1. Similarly, let $\overline{\mathcal{T}}_*^n$ denote the $\overline{\nu}_*$-dimensional space of polynomials of degree at most n which annihilate $\overline{\mathrm{L}}_*$, ie

$$\overline{\mathcal{T}}_*^n = \{\overline{R}(x) | \overline{R}(\mathrm{L}_*) = 0, \deg \overline{R} \leq n\}. \tag{3.13}$$

Arguing as above, we see that (3.12) also defines a 1-1 linear map from $\overline{\mathcal{T}}_*^n$ to V_*.

Regarded as polynomial subspaces, $\mathcal{T}_*^n$ and $\overline{\mathcal{T}}_*^n$ are transverse. For if $\sum_{j=1}^n c_j x^{j-1}$ belongs to both, then $\sum_{j=1}^n c_j \mathrm{L}^{j-1} = \sum_{j=1}^n c_j \overline{\mathrm{L}}^{j-1} = 0$. Corollary 2.1 implies that $c_2 = \cdots = c_n = 0$, which in turn implies that $c_1 = 0$. Therefore, (3.12) defines a 1-1 map from $\mathcal{T}_*^n \oplus \overline{\mathcal{T}}_*^n$ to V_*, and

$$\nu_* + \overline{\nu}_* = \dim(\mathcal{T}_*^n \oplus \overline{\mathcal{T}}_*^n) \leq \dim V_*, \tag{3.14}$$

as required. $\square$

Since we actually have an equality in (3.14), we deduce the following:

Corollary 3.1. *Given $\mathcal{T}_*^n$, $\overline{\mathcal{T}}_*^n$ and V_* as above,*

$$\sum_{j=1}^n c_j x^{j-1} \mapsto \mathbf{c} \tag{3.15}$$

is an isomorphism from $\mathcal{T}_*^n \oplus \overline{\mathcal{T}}_*^n$ to V_*.

4 Structure of singular sets

The Toda Hamiltonian has no corank-n singularities, since $dF_1 \neq 0$. The singularities of corank $(n-1)$ are relative equilibria, as is shown in the following:

Proposition 4.1. *Let $\Omega_{n-1} = \{(\mathbf{q},\mathbf{p}) |\, q_1 = \cdots = q_n, p_1 = \cdots p_n\}$ denote the set of points $(\mathbf{q},\mathbf{p})$ for which the components of $\mathbf{q}$ are all the same and the components of $\mathbf{p}$ are all the same. Then*

$$\Sigma_{n-1} = \Omega_{n-1}. \tag{4.1}$$

Proof. First we show that $\Sigma_{n-1} \subset \Omega_{n-1}$. Let $(\mathbf{q},\mathbf{p}) \in \Sigma_{n-1}$. Since $dF_1 \neq 0$, $dH(\mathbf{q},\mathbf{p})$ must be proportional to dF_1. But

$$dH = \sum_{j=1}^{n} p_j dp_j + \sum_{j=1}^{n} \left(b_j^2 - b_{j-1}^2\right) dq_j. \tag{4.2}$$

For dH to be proportional to dF_1, we must have that the p_j's are all the same and the b_j's (which are positive) are all the same. The latter implies that $q_{j+1} - q_j$ is a constant independent of j, and periodicity, ie $q_{n+1} = q_1$, then implies that the q_j's are all the same.

Next we show that $\Omega_{n-1} \subset \Sigma_{n-1}$. For $(\mathbf{q},\mathbf{p}) \in \Omega_{n-1}$, the difference equation (2.10) simplifies to

$$v_{r-1} + v_{r+1} = (\lambda - p) v_r, \tag{4.3}$$

where p is the common value of the components of $\mathbf{p}$. Periodic and antiperiodic solutions of (4.3) are given by

$$u^{\pm}_{(r)j} = \exp\left(\pm \pi i j r/n\right), \quad \lambda_r = p + 2\cos\left(\pi r/n\right), \; 0 \le r \le n, \quad r \text{ even}, \tag{4.4}$$

$$\overline{u}^{\pm}_{(s)j} = \exp\left(\pm \pi i j s/n\right), \quad \overline{\lambda}_s = p + 2\cos\left(\pi s/n\right), \; 0 < s \le n, \quad s \text{ odd}. \tag{4.5}$$

The λ_r's are doubly degenerate except for r equal to 0 and (if n is even) $n/2$, while the $\overline{\lambda}_s$'s are all doubly degenerate. Thus,

$$\nu = [\tfrac{1}{2}(n-1)], \quad \overline{\nu} = [\tfrac{1}{2}n], \tag{4.6}$$

where $[x]$ denotes the integer part of x. In general, $\nu + \overline{\nu} = n - 1$, so Theorem 3.1 implies that $(\mathbf{q},\mathbf{p}) \in \Sigma_{n-1}$. □

The explicit expressions (4.4) and (4.5) which hold in Σ_{n-1} allows us to deduce the following general result:

Proposition 4.2. *Let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\overline{\lambda}_1 \geq \cdots \geq \overline{\lambda}_n$ denote the eigenvalues of* L *and* $\overline{\mathrm{L}}$ *in descending order. Then $\lambda_r > \overline{\lambda}_r$ for r odd, $\overline{\lambda}_r > \lambda_r$ for r even, and the allowed degeneracies are $\lambda_r = \lambda_{r+1}$ for r even and $\overline{\lambda}_r = \overline{\lambda}_{r+1}$ for r odd.*

Proof. From (4.4) and (4.5), these degeneracies are simultaneously realised at the points of Σ_{n-1}. Indeed, for points in Σ_{n-1}, we have that

$$\lambda_1 > \overline{\lambda}_1 = \overline{\lambda}_2 > \lambda_2 = \lambda_3 > \cdots \overline{\lambda}_{2j-1} = \overline{\lambda}_{2j} > \lambda_{2j} = \lambda_{2j+1} > \cdots \tag{4.7}$$

(the form of the end of the sequence depends on whether n is even or odd).

For any $\mathbf{z} \in \mathbb{R}^{2n}$, (2.19) along with Newton's identities imply that the characteristic polynomials of L and $\overline{\mathrm{L}}$ differ by a constant:

$$\det(x - \mathrm{L}(\mathbf{z})) = \det(x - \overline{\mathrm{L}}(\mathbf{z})) + 4. \tag{4.8}$$

Therefore, in general, eigenvalues of L and $\overline{\mathrm{L}}$ cannot coincide. Since the eigenvalues depend continuously on $\mathbf{z}$, it follows that the inequalities in (4.7) hold not only in Σ_{n-1} but everywhere else. Therefore, in general, we have that

$$\lambda_1 > \overline{\lambda}_1 \geq \overline{\lambda}_2 > \lambda_2 \geq \lambda_3 > \cdots \overline{\lambda}_{2j-1} \geq \overline{\lambda}_{2j} > \lambda_{2j} \geq \overline{\lambda}_{2j+1} > \cdots \tag{4.9}$$

It follows that $\lambda_{2j} = \lambda_{2j+1}$ and $\overline{\lambda}_{2j-1} = \overline{\lambda}_{2j}$ are the only possible degeneracies. □

To determine the local structure of the singular set Σ, it is convenientto bring the Lax matrices L and $\overline{\mathrm{L}}$ to a canonical form. As above, let $\mathbf{z}_* \in \Sigma_k$. Let L_* denote $\mathrm{L}(\mathbf{z}_*)$, and let other functions evaluated at $\mathbf{z}_*$ be similarly denoted. Let ν_* and $\overline{\nu}_*$ denote the number of (doubly) degenerate eigenvalues of L_* and $\overline{\mathrm{L}}_*$ respectively. From Theorem 3.1, $\nu_* + \overline{\nu}_* = k$. Let λ_{r*}, $1 \leq r \leq n$ denote the eigenvalues of L_* with degenerate eigenvalues repeated, ordered so that $\lambda_{1*} = \lambda_{2*}, \ldots, \lambda_{2\nu-1*} = \lambda_{2\nu*}$. Let $\mathbf{u}_r$ denote an orthonormal set of eigenvectors of L_*. Define $\overline{\lambda}_{s*}$ and $\overline{\mathbf{u}}_s$ similarily with respect to $\overline{\mathrm{L}}_*$. Then for $\mathbf{z}$ in some neighbourhood of $\mathbf{z}_*$, there exists an orthogonal matrix $\mathrm{R}(\mathbf{z})$ depending smoothly on $\mathbf{z}$, with $\mathrm{R}_* = \mathrm{I}$, such that $\mathrm{R}^T(\mathbf{z})\mathrm{L}(\mathbf{z})\mathrm{R}(\mathbf{z})$ is block diagonal with respect to the $\mathbf{u}_r$-basis, with two-dimensional blocks for $1 \leq r \leq 2\nu_*$ and diagonal for $2\nu_* < r \leq n$. That is, letting $\Lambda(\mathbf{z})$ be the symmetric matrix with elements

$$\Lambda_{rt}(\mathbf{z}) = \mathbf{u}_r \cdot \mathrm{R}^T(\mathbf{z})\mathrm{L}(\mathbf{z})\mathrm{R}(\mathbf{z}) \cdot \mathbf{u}_t, \tag{4.10}$$

we have that

$$\Lambda(\mathbf{z}) = \begin{pmatrix} * & * & & & & & & \\ * & * & & & & & & \\ & & \ddots & & & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ & & & & & * & & \\ & & & & & & \ddots & \\ & & & & & & & * \end{pmatrix}, \tag{4.11}$$

where omitted entries are zeros. At $\mathbf{z}_*$,

$$\Lambda_* = \begin{pmatrix} \lambda_{1*} & 0 & & & & & & \\ 0 & \lambda_{1*} & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_{r*} & 0 & & & \\ & & & 0 & \lambda_{r*} & & & \\ & & & & & \lambda_{r+1*} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_{n-2\nu*} \end{pmatrix}. \tag{4.12}$$

Similarly, in some neighbourhood of $\mathbf{z}_*$, there exists an orthogonal matrix $\overline{\mathrm{R}}(\mathbf{z})$ such that $\overline{\mathrm{R}}^T(\mathbf{z})\overline{\mathrm{L}}(\mathbf{z})\overline{\mathrm{R}}(\mathbf{z})$ is block diagonal in the $\overline{\mathbf{u}}_{s*}$ basis, with two-dimensional blocks for $1 \le s \le 2\overline{\nu}_*$ and diagonal for $2\nu_* < s \le n$. Define $\overline{\Lambda}(\mathbf{z})$ in analogy with (4.10), ie

$$\overline{\Lambda}_{su}(\mathbf{z}) = \overline{\mathbf{u}}_s \cdot \mathrm{R}^T(\mathbf{z})\mathrm{L}(\mathbf{z})\mathrm{R}(\mathbf{z}) \cdot \overline{\mathbf{u}}_u. \tag{4.13}$$

Let

$$\begin{aligned} \xi_r &= \tfrac{1}{2}(\Lambda_{2r,2r} - \Lambda_{2r-1,2r-1}), & \eta_r &= \Lambda_{2r-1,2r}, & 1 &\le r \le \nu, \\ \overline{\xi}_s &= \tfrac{1}{2}(\overline{\Lambda}_{2s,2s} - \overline{\Lambda}_{2s-1,2s-1}), & \overline{\eta}_s &= \overline{\Lambda}_{2s-1,2s}, & 1 &\le s \le \overline{\nu}. \end{aligned} \tag{4.14}$$

Then, from (4.10) and (4.13), Σ_k is given locally by the vanishing of ξ_r, η_r and $\overline{\xi}_s$, $\overline{\eta}_s$, eg

$$\xi_r = \eta_r = 0, \quad 1 \le r \le \nu_*, \quad \overline{\xi}_s = \overline{\eta}_s = 0, \quad 1 \le s \le \overline{\nu}_* \tag{4.15}$$

in some neighbourhood of $\mathbf{z}_*$.

The next result, Proposition 4.3, is a general expression for Poisson brackets of spectral components of the Lax matrices evaluated at $\mathbf{z}_*$. It is used in Proposition 4.4 to show that the Poisson brackets amongst ξ_r, η_s, $\overline{\xi}_s$ and $\overline{\eta}_s$

evaluated at $\mathbf{z}_*$ are, up to normalisation, of canonical form. For the statement of Proposition 4.3, it is convenient to introduce Lax matrices in which the b_j's are allowed to have arbitrary signs. Let ϵ be an n-tuple of signs, and let L^ϵ and M^ϵ be the matrices obtained by replacing b_j with $\epsilon_j b_j$ in the expressions (2.4) for L and M. Thus, for $\epsilon = (1, \ldots, 1)$, $\mathrm{L}^\epsilon = \mathrm{L}$, while for $\epsilon = (1, \ldots, 1, -1)$, $\mathrm{L}^\epsilon = \overline{\mathrm{L}}$. We regard ϵ as n-periodic in its index.

Proposition 4.3. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *denote eigenvectors of* L^ϵ_* *with the same eigenvalue* λ*, ie*

$$\mathrm{L}^\epsilon_* \cdot \mathbf{u} = \lambda \mathbf{u}, \quad \mathrm{L}^\epsilon_* \cdot \mathbf{v} = \lambda \mathbf{v}, \tag{4.16}$$

where $\mathbf{u}$ *and* $\mathbf{v}$ *need not be linearly independent. Similarly, let* $\mathbf{w}$ *and* $\mathbf{x}$ *denote eigenvectors of* L^σ *with the same eigenvalue* μ*,*

$$\mathrm{L}^\sigma_* \cdot \mathbf{w} = \mu \mathbf{w}, \quad \mathrm{L}^\sigma_* \cdot \mathbf{x} = \mu \mathbf{x}. \tag{4.17}$$

Let

$$\begin{aligned} L^\epsilon_{uv} &= \mathbf{u} \cdot \mathrm{L}^\epsilon \cdot \mathbf{v}, \\ L^\sigma_{wx} &= \mathbf{w} \cdot \mathrm{L}^\epsilon \cdot \mathbf{x}. \end{aligned} \tag{4.18}$$

Then if $\lambda = \mu$ *and* $\prod_{m=1}^n \epsilon_m \sigma_m = 1$*,*

$$\{L^\epsilon_{uv}, L^\sigma_{wx}\}_* = \frac{1}{n}\left((\mathbf{v} \cdot \mathrm{D} \cdot \mathbf{x})(\mathbf{u} \cdot \mathrm{M}^\epsilon_* \mathrm{D} \cdot \mathbf{w}) + (\mathbf{u} \cdot \mathrm{D} \cdot \mathbf{w})(\mathbf{v} \cdot \mathrm{M}^\epsilon_* \mathrm{D} \cdot \mathbf{x})\right), \tag{4.19}$$

where D *is the diagonal matrix with diagonal elements*

$$d_m = \epsilon_1 \sigma_1 \cdots \epsilon_{m-1} \sigma_{m-1}. \tag{4.20}$$

Otherwise, ie if $\lambda \neq \mu$ *or* $\prod_{m=1}^n \epsilon_m \sigma_m = -1$*, then*

$$\{L^\epsilon_{uv}, L^\sigma_{wx}\}_* = 0. \tag{4.21}$$

We note that $\prod_{m=1}^n \epsilon_m \sigma_m = 1$ if and only if L^ϵ and L^σ are conjugate. In fact, we shall only use the cases where L^ϵ and L^σ are (independently) either L or $\overline{\mathrm{L}}$.

Proof. Using (2.4) and (2.23), it is straightforward to show that

$$\begin{aligned} \{L^\epsilon_{uv}, L^\sigma_{wx}\} =& \tfrac{1}{2}\sum_{m=1}^n v_m x_m (A_m(\mathbf{u}, \mathbf{w}) + \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{u}, \mathbf{w})) + \\ & \tfrac{1}{2}\sum_{m=1}^n u_m w_m (A_m(\mathbf{v}, \mathbf{x}) + \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{x}, \mathbf{v})), \end{aligned} \tag{4.22}$$

where

$$
\begin{aligned}
A_m(\mathbf{u}, \mathbf{w}) &= b_m(\epsilon_m u_{m+1} w_m - \sigma_m u_m w_{m+1}), \\
A_m(\mathbf{v}, \mathbf{x}) &= b_m(\epsilon_m v_{m+1} x_m - \sigma_m v_m x_{m+1}). \qquad (4.23)
\end{aligned}
$$

Indeed, (4.22) holds independently of the eigenvector equations (4.16) and (4.17). The eigenvector equations imply additionally that, at $\mathbf{z}_*$, $A_{m*}(\mathbf{u}, \mathbf{w})$ and $A_{m*}(\mathbf{v}, \mathbf{x})$ satisfy a Wronskian-like first-order difference equation. Indeed, (4.16) and (4.17) yield second-order difference equations for $\mathbf{u}$ and $\mathbf{w}$ (cf (2.10)),

$$
(\epsilon_m b_m u_{m+1} + p_m u_m + \epsilon_{m-1} b_{m-1} u_{m-1})_* = \lambda u_m, \qquad (4.24a)
$$

$$
(\sigma_m b_m w_{m+1} + p_m w_m + \sigma_{m-1} b_{m-1} w_{m-1})_* = \mu w_m. \qquad (4.24b)
$$

Multiplying (4.24a) by w_m and (4.24b) by u_m, and subtracting, we get

$$
A_{m*}(\mathbf{u}, \mathbf{w}) - \epsilon_{m-1}\sigma_{m-1} A_{m-1*}(\mathbf{u}, \mathbf{w}) = (\lambda - \mu) u_m w_m. \qquad (4.25a)
$$

Similarly,

$$
A_{m*}(\mathbf{v}, \mathbf{x}) - \epsilon_{m-1}\sigma_{m-1} A_{m-1*}(\mathbf{v}, \mathbf{x}) = (\lambda - \mu) v_m x_m. \qquad (4.25b)
$$

Suppose that $\lambda \neq \mu$. From (4.25), we have that

$$
u_m w_m = \frac{1}{\Delta}(A_m(\mathbf{u}, \mathbf{w}) - \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{u}, \mathbf{w}))_*, \qquad (4.26a)
$$

$$
v_m x_m = \frac{1}{\Delta}(A_m(\mathbf{v}, \mathbf{x}) - \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{v}, \mathbf{x}))_*, \qquad (4.26b)
$$

where $\Delta = \lambda - \mu$. Substituting into (4.22), we get that

$$
\begin{aligned}
\{L^\epsilon_{uv}, L^\sigma_{wx}\}_* &= \frac{1}{2\Delta} \sum_{m=1}^{n} \Big[\\
&(A_m(\mathbf{v}, \mathbf{x}) - \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{v}, \mathbf{x})) \times (A_m(\mathbf{u}, \mathbf{w}) + \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{u}, \mathbf{w})) + \\
&(A_m(\mathbf{u}, \mathbf{w}) - \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{u}, \mathbf{w})) \times (A_m(\mathbf{v}, \mathbf{x}) + \epsilon_{m-1}\sigma_{m-1} A_{m-1}(\mathbf{v}, \mathbf{x})) \Big]_* \\
&= \frac{1}{\Delta} \sum_{m=1}^{n} \Big[A_m(\mathbf{u}, \mathbf{w}) A_m(\mathbf{v}, \mathbf{x}) - A_{m-1}(\mathbf{u}, \mathbf{w}) A_{m-1}(\mathbf{v}, \mathbf{x}) \Big]_* = 0, \qquad (4.27)
\end{aligned}
$$

as required by (4.21).

Next, suppose that $\lambda = \mu$. From (4.25),

$$
A_{m*}(\mathbf{u}, \mathbf{w}) = \epsilon_{m-1}\sigma_{m-1} A_{m-1*}(\mathbf{u}, \mathbf{w}), \qquad (4.28a)
$$

$$
A_{m*}(\mathbf{v}, \mathbf{x}) = \epsilon_{m-1}\sigma_{m-1} A_{m-1*}(\mathbf{v}, \mathbf{x}). \qquad (4.28b)
$$

Substituting into (4.22), we get that

$$\{L^\epsilon_{uv}, L^\sigma_{wx}\}_* = \sum_{m=1}^n v_m x_m A_{m*}(\mathbf{u},\mathbf{w}) + \sum_{m=1}^n u_m w_m A_{m*}(\mathbf{v},\mathbf{x}). \tag{4.29}$$

Iterating the relations (4.28) n times, we get that $A_{m*}(\mathbf{u},\mathbf{w}) = \epsilon_1\sigma_1\cdots\epsilon_n\sigma_n A_{m*}(\mathbf{u},\mathbf{w})$ and similarly for $A_{m*}(\mathbf{v},\mathbf{x})$. Therefore, if $\epsilon_1\sigma_1\cdots\epsilon_n\sigma_n = -1$, $A_{m*}(\mathbf{u},\mathbf{w}) = A_{m*}(\mathbf{v},\mathbf{x}) = 0$, so that $\{L^\epsilon_{uv}, L^\sigma_{wx}\}_* = 0$, as in (4.21).

Now suppose that $\epsilon_1\sigma_1\cdots\epsilon_n\sigma_n = 1$. In this case, (4.28a) implies that

$$d_m A_{m*}(\mathbf{u},\mathbf{w}) = d_{m-1}A_{m-1*}(\mathbf{u},\mathbf{w}), \tag{4.30}$$

where d_m is given by (4.20), ie, $d_m A_m(\mathbf{u},\mathbf{w})$ is independent of m. Thus, the first term in (4.29) can be expressed as

$$\begin{aligned}\sum_{m=1}^n v_m x_m A_{m*}(\mathbf{u},\mathbf{w}) &= \left(\sum_{m=1}^n d_m v_m x_m\right)\left(\frac{1}{n}\sum_{m=1}^n d_m A_{m*}(\mathbf{u},\mathbf{w})\right)\\ &= (\mathbf{v}\cdot \mathrm{D}\cdot \mathbf{x})\left(\frac{1}{n}\sum_{m=1}^n d_m A_{m*}(\mathbf{u},\mathbf{w})\right).\end{aligned} \tag{4.31}$$

The sum in the last term in (4.31) can be expressed as

$$\begin{aligned}\sum_{m=1}^n d_m A_{m*}(\mathbf{u},\mathbf{w}) &= \sum_{m=1}^n d_m b_{m*}(\epsilon_m u_{m+1}w_m - \sigma_m u_m w_{m+1}) =\\ &\sum_{m=1}^n \epsilon_m b_{m*}(u_{m+1}(\mathrm{D}\cdot\mathbf{w})_m - u_m(\mathrm{D}\cdot\mathbf{w})_{m+1}) = \mathbf{u}\cdot \mathrm{M}^\epsilon_* \mathrm{D}\cdot\mathbf{w}.\end{aligned} \tag{4.32}$$

Substituting (4.31) and (4.32) into the first term of (4.29) and making analogous substitutions for the second term, we get

$$\{L^\epsilon_{uv}, L^\sigma_{wx}\}_* = \frac{1}{n}(\mathbf{v}\cdot\mathrm{D}\cdot\mathbf{x})(\mathbf{u}\cdot\mathrm{M}^\epsilon_*\mathrm{D}\cdot\mathbf{w}) + \frac{1}{n}(\mathbf{u}\cdot\mathrm{D}\cdot\mathbf{w})(\mathbf{v}\cdot\mathrm{M}^\epsilon_*\mathrm{D}\cdot\mathbf{x}), \tag{4.33}$$

as in (4.19). □

Proposition 4.4. *Let ξ_r, η_r, $\overline{\xi}_s$ and $\overline{\eta}_s$ be given by (4.14). Then*

$$\{\xi_r, \overline{\xi}_s\}_* = \{\xi_r, \overline{\eta}_s\}_* = \{\eta_r, \overline{\xi}_s\}_* = \{\eta_r, \overline{\eta}_s\}_* = 0, \tag{4.34a}$$

$$\{\xi_r, \xi_t\}_* = \{\eta_r, \eta_t\}_* = \{\overline{\xi}_s, \overline{\xi}_u\}_* = \{\overline{\eta}_s, \overline{\eta}_u\}_* = 0, \tag{4.34b}$$

$$\{\xi_r, \eta_t\}_* = \frac{1}{n}\mathbf{u}_{2r}\cdot\mathrm{M}_*\cdot\mathbf{u}_{2r-1}\,\delta_{rt}, \quad \{\overline{\xi}_s, \overline{\eta}_u\}_* = \frac{1}{n}\overline{\mathbf{u}}_{2s}\cdot\overline{\mathrm{M}}_*\cdot\overline{\mathbf{u}}_{2s-1}\,\delta_{su}. \tag{4.34c}$$

Moreover, $\mathbf{u}_{2r}\cdot\mathrm{M}_*\cdot\mathbf{u}_{2r-1}$ *and* $\overline{\mathbf{u}}_{2s}\cdot\overline{\mathrm{M}}_*\cdot\overline{\mathbf{u}}_{2s-1}$ *do not vanish.*

Proof. We treat a representative case for each set of relations in (4.34). For (4.34a), we consider $\{\eta_r, \overline{\xi}_s\}_*$. From (4.14) and (4.10),

$$d\eta_{r*} = d(\mathbf{u}_{2r-1} \cdot \mathrm{R}^T\mathrm{LR} \cdot \mathbf{u}_{2r})_* = \mathbf{u}_{2r-1} \cdot [\mathrm{L}, d\mathrm{R}]_* \cdot \mathbf{u}_{2r} + \mathbf{u}_{2r-1} \cdot d\mathrm{L}_* \cdot \mathbf{u}_{2r}$$
$$= \mathbf{u}_{2r-1} \cdot d\mathrm{L}_* \cdot \mathbf{u}_{2r}, \quad (4.35)$$

where we have used the fact that $\mathrm{R}_* = \mathrm{I}$, $d\mathrm{R}_*$ is antisymmetric (since R is orthogonal) and, in the last equality, that $\mathbf{u}_{2r-1}$ and $\mathbf{u}_r$ are eigenvectors of L_* with the same eigenvalue . Similarly, from (4.14) and (4.13),

$$d\overline{\xi}_{s*} = \tfrac{1}{2}\overline{\mathbf{u}}_{2s} \cdot d\overline{\mathrm{L}}_* \cdot \overline{\mathbf{u}}_{2s} - \tfrac{1}{2}\overline{\mathbf{u}}_{2s-1} \cdot d\overline{\mathrm{L}}_* \cdot \overline{\mathbf{u}}_{2s-1}. \quad (4.36)$$

Together, (4.35) and (4.36) imply that

$$\{\eta_r, \overline{\xi}_s\}_* = \{\mathbf{u}_{2r-1} \cdot \mathrm{L} \cdot \mathbf{u}_{2r}, \tfrac{1}{2}\overline{\mathbf{u}}_{2s} \cdot \overline{\mathrm{L}} \cdot \overline{\mathbf{u}}_{2s} - \tfrac{1}{2}\overline{\mathbf{u}}_{2s-1} \cdot \overline{\mathrm{L}} \cdot \overline{\mathbf{u}}_{2s-1}\}_*. \quad (4.37)$$

Proposition 4.3 then implies, with $\epsilon = (1, \dots, 1)$ and $\sigma = (1, \dots, -1)$, that

$$\{\eta_r, \overline{\xi}_s\}_* = 0, \quad (4.38)$$

since $\prod_{m=1}^n \epsilon_m \sigma_m = -1$. The remaining relations in (4.34a) are obtained similarly.

For (4.34b), we consider $\{\eta_r, \eta_t\}_*$. If $r = t$ the bracket obviously vanishes, so we take $r \neq t$. From (4.35) it follows that

$$\{\eta_r, \eta_t\}_* = \{\mathbf{u}_{2r-1} \cdot \mathrm{L} \cdot \mathbf{u}_{2r}, \mathbf{u}_{2t-1} \cdot \mathrm{L} \cdot \mathbf{u}_{2t}\}_*. \quad (4.39)$$

Since $\lambda_r \neq \lambda_t$, Proposition 4.3 implies that $\{\eta_r, \eta_t\}_* = 0$. The remaining relations in (4.34b) are obtained similarly.

For (4.34c), we consider $\{\xi_r, \eta_t\}_*$. From (4.35) and (4.36) (or rather, its analog for $d\xi_r$), we get that

$$\{\xi_r, \eta_t\}_* = \tfrac{1}{2}\{\mathbf{u}_{2r} \cdot \mathrm{L} \cdot \mathbf{u}_{2r}, \mathbf{u}_{2t-1} \cdot \mathrm{L} \cdot \mathbf{u}_{2t}\}_* - \tfrac{1}{2}\{\mathbf{u}_{2r-1} \cdot \mathrm{L} \cdot \mathbf{u}_{2r-1}, \mathbf{u}_{2t-1} \cdot \mathrm{L} \cdot \mathbf{u}_{2t}\}_*. \quad (4.40)$$

Proposition 4.3 then implies, with $\epsilon = \sigma = (1, \dots, 1)$, that (4.40) vanishes if $r \neq t$ (as, in this case, $\lambda_r \neq \lambda_t$). On the other hand, for $r = t$, Proposition 4.3, with $\mathrm{D} = \mathrm{I}$, and $\mathbf{u}_{2r-1}$, $\mathbf{u}_{2r}$ orthonormal gives that

$$\{\xi_r, \eta_r\}_* = \frac{1}{2n}\mathbf{u}_{2r} \cdot \mathrm{M}_* \cdot \mathbf{u}_{2r-1} - \frac{1}{2n}\mathbf{u}_{2r-1} \cdot \mathrm{M}_* \cdot \mathbf{u}_{2r} = \frac{1}{n}\mathbf{u}_{2r-1} \cdot \mathrm{M}_* \cdot \mathbf{u}_{2r}. \quad (4.41)$$

The quantity $\mathbf{u}_{2r} \cdot \mathrm{M}_* \cdot \mathbf{u}_{2r-1}$ which appears in (4.41) does not vanish. This is because, from (4.30) and (4.32) (with $d_m = 1$), we have that $\mathbf{u}_{2r} \cdot \mathrm{M}_* \cdot \mathbf{u}_{2r-1}$ is equal to $nb_m(u_{2r-1,m+1}u_{2r,m} - u_{2r-1,m}u_{2r,m+1})$ for all m. $\mathbf{u}_{2r} \cdot \mathrm{M}_* \cdot \mathbf{u}_{2r-1} = 0$ would imply that $\mathbf{u}_{2r-1}$ and $\mathbf{u}_{2r}$ are proportional, contradicting orthonormality. The result for $\{\overline{\xi}_s, \overline{\eta}_u\}_*$ follows similarly. $\square$

Theorem 4.1. *Σ_k is a codimension-$2k$ symplectic submanifold. It consists of $\binom{n-1}{k}$ components which are disconnected from each other. Σ_k is contained in the closure of Σ_j for all $j < k$.*

Proof. Let $\mathbf{z}_* \in \Sigma_k$ and let ν_*, $\overline{\nu}_*$ denote the number of degenerate eigenvalues of L_* and $\overline{\mathrm{L}}_*$, respectively, so that, by Theorem 3.1, $\nu_* + \overline{\nu}_* = k$. Let ξ_r, η_r, $1 \le r \le \nu_*$, and $\overline{\xi}_s$, $\overline{\eta}_s$, $1 \le s \le \overline{\nu}_*$, be given by (4.14) in some neighbourhood of $\mathbf{z}_*$. In this neighbourhood, Σ_k is given by

$$\xi_r = \eta_r = \overline{\xi}_s = \overline{\eta}_s = 0. \tag{4.42}$$

Proposition 4.4 implies that the derivatives of ξ_r, η_r, $\overline{\xi}_s$ and $\overline{\eta}_s$ are linearly independent at $\mathbf{z}_*$. It follows from the implicit function theorem that Σ_k is codimension-$2k$ submanifold.

Next, we show that the tangent space of $\Sigma_k \subset \mathbb{R}^{2n}$ at $\mathbf{z}_*$ is symplectic. Let $\mathbf{X}_* \in T_{\mathbf{z}_*}\Sigma_k$. $\mathbf{X}_*$ may be extended to a vector field $\mathbf{X}$ defined in a neighbourhood of $\mathbf{z}_*$ whose restriction to Σ_k is tangent to Σ_k. Then (4.42) implies, for example, that $(\mathbf{X}\cdot d\xi_r)_* = 0$. But $(\mathbf{X}\cdot d\xi_r)_*$ is just the symplectic inner product of $\mathbf{X}_*$ with the Hamiltonian vector field generated by ξ_r at $\mathbf{z}_*$. Let E_* denote the subspace of $T_*\mathbb{R}^{2n}$ spanned by the Hamiltonian vector fields generated by ξ_r, η_r, and $\overline{\xi}_s$, $\overline{\eta}_s$. Arguing as above, we may conclude that $\mathbf{X}_*$, and therefore $T_*\Sigma_k$, is skew-orthogonal to E_*. Proposition 4.4 implies that E_* is symplectic of dimension $2k$. Since $\operatorname{codim} T_{\mathbf{z}_*}\Sigma_k = 2k$, it follows that $T_{\mathbf{z}_*}\Sigma_k$ is the skew-orthogonal complement of E_*. The skew-orthogonal complement of a symplectic subspace is itself symplectic, so $T_{\mathbf{z}_*}\Sigma_k$ is symplectic, and therefore Σ_k is a symplectic submanifold.

Σ_k can be partitioned into distinct components according to the particular k pairs of eigenvalues which are degenerate. From Proposition 4.2 there are $(n-1)$ possible degeneracies ($[\frac{1}{2}(n-1)]$ from L and $[\frac{1}{2}n]$ from $\overline{\mathrm{L}}$), so the maximum number of components is $\binom{n-1}{k}$. Every choice of k pairs can be realised by setting, in a neighbourhood of a point of Σ_{n-1}, the appropriate pairs of coordinates (ξ_r, η_r), $(\overline{\xi}_s, \overline{\eta}_s)$ to be nonzero, and the remaining pairs to be nonzero. Thus Σ_k has precisely $\binom{n-1}{k}$ components. Along any path connecting points in different components, the number of eigenvalue degeneracies must change, so the path must leave Σ_k. Therefore, the components are disconnected from each other.

Any neighbourhood of $\mathbf{z}_* \in \Sigma_k$ contains points of $\Sigma_{j<k}$. Such points are obtained by setting $k-j$ of the k coordinates pairs (ξ_r, η_r) and $(\overline{\xi}_s, \overline{\eta}_s)$ to be nonzero. Therefore, Σ_k is contained in the closure of $\Sigma_{j<k}$. ☐

We consider next the transverse stability of Σ_k. Let $\mathbf{z}_* \in \Sigma_k$. In analogy

with (4.14), we define the local functions

$$\begin{aligned}
\tau_r &= \tfrac{1}{2}(\Lambda_{2r,2r} + \Lambda_{2r-1,2r-1}), && 1 \le r \le \nu, \\
\overline{\tau}_s &= \tfrac{1}{2}(\overline{\Lambda}_{2s,2s} + \overline{\Lambda}_{2s-1,2s-1}), && 1 \le s \le \overline{\nu},
\end{aligned} \tag{4.43}$$

with

$$\begin{aligned}
\tau_{r*} &= \tfrac{1}{2}\mathbf{u}_{2r} \cdot d\mathrm{L}_* \cdot \mathbf{u}_{2r} + \tfrac{1}{2}\mathbf{u}_{2r-1} \cdot d\mathrm{L}_* \cdot \mathbf{u}_{2r-1}, && 1 \le r \le \nu, \\
\overline{\tau}_{s*} &= \tfrac{1}{2}\overline{\mathbf{u}}_{2s} \cdot d\overline{\mathrm{L}}_* \cdot \overline{\mathbf{u}}_{2s} + \tfrac{1}{2}\overline{\mathbf{u}}_{2s-1} \cdot d\overline{\mathrm{L}}_* \cdot \overline{\mathbf{u}}_{2s-1}, && 1 \le s \le \overline{\nu}.
\end{aligned} \tag{4.44}$$

With arguments similar to those of Proposition 4.4, one can show that the Poisson brackets of the τ_r's and the $\overline{\tau}_s$'s with ξ_r, η_r, $\overline{\xi}_s$ and $\overline{\eta}_s$ all vanish at $\mathbf{z}_*$. Likewise, the Poisson brackets of the τ_r's and $\overline{\tau}_s$'s amongst themselves vanish at $\mathbf{z}_*$. We record this briefly as

$$\{\tau_{\cdot}, \chi_{\cdot}\}_* = \{\tau_{\cdot}, \tau_{\cdot\cdot}\}_* = 0, \tag{4.45}$$

where $\chi_{\cdot}$ denotes ξ_r, η_r, $\overline{\xi}_s$ or $\overline{\eta}_s$, and $\tau_{\cdot}$, $\tau_{\cdot\cdot}$ denotes τ_r or $\overline{\tau}_s$.

Let

$$\begin{aligned}
T_r(x) &= \frac{\det(\mathrm{L}_* - x\mathrm{I})}{\lambda_{r*} - x} = \sum_{j=1}^{n} c_{rj} x^{j-1}, && 1 \le r \le \nu, \\
\overline{T}_{s*}(x) &= \frac{\det(\overline{\mathrm{L}}_* - x\mathrm{I})}{\overline{\lambda}_{s*} - x} = \sum_{j=1}^{n} \overline{c}_{sj} x^{j-1}, && 1 \le s \le \overline{\nu}.
\end{aligned} \tag{4.46}$$

Clearly $T_r(\mathrm{L}_*) = 0$, and moreover, the T_r's are linearly independent (since, if $\sum_{t=1}^{\nu} a_t T_t(\lambda_{r*}) = 0$, then $\sum_{t=1}^{\nu} a_t T_t'(\lambda_{r*}) = 0$, and the fact that $T_t'(\lambda_{r_*}) = 0$ for $r \ne t$ implies that $a_r = 0$ for each r). Therefore, the T_r's constitute a basis for $\mathcal{T}_*^n$, the space of polynomials of degree at most n that annihilate L_* (cf (3.10)). Likewise, the $\overline{T}_s$'s constitute a basis for $\overline{\mathcal{T}}_*^n$, the space of polynomials of degree at most n that annihilate $\overline{\mathrm{L}}_*$ (cf (3.13)). It follows from Corollary 3.1 that the vectors $\mathbf{c}_r = (c_{r1}, \ldots, c_{rn})$ and $\overline{\mathbf{c}}_s = (\overline{c}_{s1}, \ldots, \overline{c}_{sn})$ constitute a basis for V_*, the space of linear relations amongst the dF_{j*}'s. Therefore, letting

$$G_r = \sum_{j=1}^{n} c_{rj} F_j, \quad \overline{G}_s = \sum_{j=1}^{n} \overline{c}_{sj} F_j, \tag{4.47}$$

we have that the Hamiltonian flows generated by G_r and $\overline{G}_s$ have $\mathbf{z}_*$ as a fixed point. The stability of these flows at $\mathbf{z}_*$ is given by the following:

Theorem 4.2.

$$\begin{aligned} G''_{r*} &= 2T'_r(\lambda_{r*})\,(d\xi_r\otimes d\xi_r + d\eta_r\otimes d\eta_r + d\tau_r\otimes d\tau_r)_*\,,\\ \overline{G}''_{s*} &= 2\overline{T}'_s(\overline{\lambda}_{s*})\left(d\overline{\xi}_s\otimes d\overline{\xi}_s + d\overline{\eta}_s\otimes d\overline{\eta}_s + d\overline{\tau}_s\otimes d\overline{\tau}_s\right)_*. \end{aligned} \tag{4.48}$$

Thus, the linearised G_r-flow about $\mathbf{z}_*$ produces elliptic oscillations in the (ξ_r,η_r) plane with frequencies

$$\omega_r = 2nT'(\lambda_{r*})/(\mathbf{u}_{2r}\cdot \mathrm{M}_*\cdot \mathbf{u}_{2r-1}). \tag{4.49}$$

Similarly, the linearised $\overline{G}_s$-flow about $\mathbf{z}_*$ produces elliptic oscillations in the $(\overline{\xi}_s,\overline{\eta}_s)$ plane with frequencies

$$\overline{\omega}_s = 2n\overline{T}'(\overline{\lambda}_{s*})/(\overline{\mathbf{u}}_{2s}\cdot \overline{\mathrm{M}}_*\cdot \overline{\mathbf{u}}_{2s-1}). \tag{4.50}$$

Proof. We carry out the calculations for G_r; those for $\overline{G}_s$ are similar. From (4.47) and (2.5), we have that

$$G''_{r*} = \sum_{j=1}^{n}\mathrm{Tr}\,(c_{jr}\mathrm{L}_*^{j-1}\mathrm{L}''_*) + \sum_{j=1}^{n} c_{jr}\mathrm{Tr}\,(d\mathrm{L}^{j-1}\otimes d\mathrm{L})_*. \tag{4.51}$$

As $T_r(\mathrm{L}_*)=0$, the first term vanishes. As for the second term,

$$\begin{aligned} \sum_{j=1}^{n} c_{jr}\mathrm{Tr}\,(d\mathrm{L}^{j-1}\otimes d\mathrm{L})_* &= \sum_{j=1}^{n}\sum_{k=0}^{j-2} c_{jr}\mathrm{Tr}\,(\mathrm{L}_*^k d\mathrm{L}_*\mathrm{L}_*^{j-k-2}\otimes d\mathrm{L}_*)\\ &= \sum_{a=1}^{n-\nu}\sum_{b=1}^{n-\nu}\sum_{j=1}^{n}\sum_{k=0}^{j-2} c_{jr}\lambda_{a*}^k\lambda_{b*}^{j-k-2}\mathrm{Tr}\,(\rho_{a*}d\mathrm{L}_*\otimes\rho_{b*}d\mathrm{L}_*), \end{aligned} \tag{4.52}$$

where we have introduced the spectral resolution of L_*,

$$\mathrm{L}_* = \sum_{a=1}^{n-\nu}\lambda_{a*}\rho_{a*},\quad \rho_{a*} = \frac{\prod_{r\neq a}(\mathrm{L}_*-\lambda_{a*})}{\prod_{r\neq a}(\lambda_{r*}-\lambda_{a*})}. \tag{4.53}$$

ρ_{a*} is the symmetric projector onto the λ_{a*}-eigenspace of L_{a*}, so that $\rho_{a*}\rho_{b*} = \delta_{ab}\rho_{a*}$. The sums over j and k in (4.52) are readily performed to give

$$\sum_{j=1}^{n}\sum_{k=0}^{j-2} c_{jr}\lambda_{a*}^k\lambda_{b*}^{j-k-2} = \begin{cases}(T_r(\lambda_{a*})-T_r(\lambda_{b*}))/(\lambda_{a*}-\lambda_{b*}), & a\neq b,\\ T'_r(\lambda_{a*}), & a=b.\end{cases} \tag{4.54}$$

We have that $T_r(\lambda_{a*}) = 0$ for all a, while $T_r'(\lambda_{a*})$ vanishes unless $a = r$. Thus, the sums over a and b in (4.52) collapse to the single term $a = b = r$. Substituting into (4.51), we get

$$G_{r*}'' = T_r'(\lambda_{r*})\mathrm{Tr}\,(\rho_{r*}d\mathrm{L}_* \otimes \rho_{r*}d\mathrm{L}_*). \tag{4.55}$$

The projector ρ_{r*} may be written as the diadic $\mathbf{u}_{2r-1}\mathbf{u}_{2r-1} + \mathbf{u}_{2r}\mathbf{u}_{2r}$. Substituting into (4.55), we get

$$\begin{aligned} G_{r*}'' = T_r'(\lambda_{r*})\,[&(\mathbf{u}_{2r}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r})\otimes(\mathbf{u}_{2r}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r}) \\ &+ 2(\mathbf{u}_{2r-1}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r})\otimes(\mathbf{u}_{2r-1}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r}) \\ &+(\mathbf{u}_{2r-1}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r-1})\otimes(\mathbf{u}_{2r-1}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r-1})]\,. \end{aligned} \tag{4.56}$$

Since (cf (4.35), (4.36), (4.44))

$$\begin{aligned} d\xi_{r*} + d\tau_{r*} &= \mathbf{u}_{2r}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r}, \\ d\xi_{r*} - d\tau_{r*} &= \mathbf{u}_{2r-1}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r-1}, \\ d\eta_{r*} &= \mathbf{u}_{2r-1}\cdot d\mathrm{L}_*\cdot\mathbf{u}_{2r}, \end{aligned} \tag{4.57}$$

(4.56) yields the required result (4.48).

(4.48), along with the Poisson bracket relations of Proposition 4.4 and those recorded in (4.45), imply that the equations of motion for the linearised G_r-flow about $\mathbf{z}_*$ transverse to Σ_k are given by

$$\dot{\xi}_r = \omega_r\eta_r, \quad \dot{\eta}_r = -\omega_r\xi_r, \tag{4.58}$$

while the remaining transverse coordinates $\xi_{u\neq r}, \eta_{u\neq r}, \overline{\xi}_s, \overline{\eta}_s$ are fixed. Thus, the linearised transverse G_r-flow is elliptic in the (ξ_r, η_r)-plane with frequency ω_r. $\square$

5 Maslov indices and eigenvector monodromies.

Consider an integrable system in $\mathbb{R}^{2n}$ with integrals of motion $F : \mathbb{R}^{2n} \to \mathbb{R}^n$. In the set of regular points R of F, the Hamiltonian vector fields generated by the F_j's span a n-dimensional Lagrangian plane, which we denote $\lambda(\mathbf{z})$. As shown by Arnold [4], the space of n-dimensional Lagrangian planes, $\Lambda(n)$, has fundamental group $\pi_1(\Lambda(n)) = \mathbb{Z}$, and the Maslov index of a continuous, oriented closed curve C in R is the degree (winding number) of $\lambda(C)$ in $\Lambda(n)$,

$$\mu(C) = \mathrm{wn}\,\lambda(C). \tag{5.1}$$

For angle contours C_j on an invariant torus, the Maslov index appears in the semiclassical EBK quantisation conditions for the associated action variables [21, 20],

$$I_j = (n_j + \tfrac{1}{4}\mu_j)\hbar. \tag{5.2}$$

In this case $\mu(C)$ is always even (this is because the distribution of Lagrangian planes $\lambda(\mathbf{z})$ over R is orientable).

In a companion paper [16], we show that under certain genericity conditions, the Maslov index of C can be expressed as a sum of contributions from the nondegenerate corank-one singularities it encloses. We briefly summarise the results. Let Σ denote the critical set of F, and let

$$\Sigma_k = \{\mathbf{z} \in \mathbb{R}^{2n} |\, \operatorname{corank} dF(\mathbf{z}) = k\} \tag{5.3}$$

denote the set of critical points of F of corank k. Given $\mathbf{z}_* \in \Sigma_1$, let $\mathbf{c} \in \mathbb{R}^n$ be a nontrivial solution of $\sum_{j=1}^n c_j dF_{j*} = 0$ (here $dF_* = dF(\mathbf{z}_*)$). Let $K_* = \mathrm{J}\left(\sum_{j=1}^n c_j \mathrm{F}''_{j*}\right)$. We say that $\mathbf{z}_*$ is *nondegenerate* if $\operatorname{Tr} \mathrm{K}_*^2 \neq 0$. Let Δ denote the set of nondegenerate points in Σ_1,

$$\Delta = \{\mathbf{z}_* \in \Sigma_1 |\, \operatorname{Tr} \mathrm{K}_*^2 \neq 0\}. \tag{5.4}$$

In general, Δ is a codimension-2 symplectic submanifold. Let $S : D^2 \to \mathbb{R}^{2n}$ denote a map of the oriented unit two-disk D^2 into $\mathbb{R}^{2n}$ smooth on the interior of D^2 such that $S|_{\partial D^2} = C$. We assume that S is *transverse* to Σ. By this we mean that i) the only critical points of dF contained in the image of S belong to Δ, ii) $S^{-1}(\Delta)$ consists of a finite set of points $e_1, \ldots, e_r$ in the interior of D^2, and iii) $dS(e_j)$ has full rank. (If $\Sigma - \Sigma_1$ is composed of submanifolds of codimension three or more, such an S can always be found.) Let $\mathbf{z}_j = S(e_j)$ denote the critical points of F in the image of S. Let $E_{\mathbf{z}_j}$ denote the skew-orthogonal complement of $T_{\mathbf{z}_j}\Delta$. $E_{\mathbf{z}_j}$ is a two-dimensional symplectic plane. Let $P_j : T_{\mathbf{z}_j}\mathbb{R}^{2n} \to E_{\mathbf{z}_j}$ denote the projection onto $E_{\mathbf{z}_j}$ with respect to the decomposition

$$T_{\mathbf{z}_j}\mathbb{R}^{2n} = T_{\mathbf{z}_j}\Delta \oplus E_{\mathbf{z}_j}. \tag{5.5}$$

Then the map $P_j \circ dS : T_{e_j} D^2 \to E_{\mathbf{z}_j}$ is nonsingular. Let $\sigma_j = \pm 1$ according to whether this map is orientation-preserving or reversing (the orientation on $E_{\mathbf{z}_j}$ is given by the symplectic form). Then

$$\mu(C) = 2 \sum_j \sigma_j \cdot \operatorname{sgn} \operatorname{Tr} \mathrm{K}_{\mathbf{z}_j}^2. \tag{5.6}$$

For the periodic Toda chain, all critical points are nondegenerate of elliptic type (cf Theorem 4.2), so that $\operatorname{Tr} \mathrm{K}_{\mathbf{z}_j}^2 < 0$. Also, $\Sigma - \Sigma_1 = \cup_{k>1}\Sigma_k$

is composed of submanifolds of codimension four or more (Theorem 4.1). Thus, for C a continuous closed curve in the regular component of the Toda integrals of motion, we have that

$$\mu(C) = -2 \sum_j \sigma_j. \tag{5.7}$$

We can also associate to C the signs acquired by the normalised eigenvectors of L and $\overline{\mathrm{L}}$ on continuation around C. Explicitly, let the eigenvalues of L and $\overline{\mathrm{L}}$ be indexed in increasing order, so that $\lambda_1 < \cdots < \lambda_n$ and $\overline{\lambda}_1 < \cdots < \overline{\lambda}_n$ in R. Let $\mathbf{z}(t)$, $0 \le t \le 1$ denote a parameterisation of C, and let $\mathbf{u}_r(t)$ denote a continuously varying, normalised real eigenvector of $\mathrm{L}(\mathbf{z}(t))$ with eigenvalue $\lambda_r(\mathbf{z}(t))$. Then

$$\mathbf{u}_r(1) = \gamma_r(C)\mathbf{u}_r(0), \tag{5.8}$$

where $\gamma_r(C) = \pm 1$. Defining $\overline{\mathbf{u}}_s(t)$ analogously to be a continuously varying, normalised eigenvector of $\overline{\mathrm{L}}(\mathbf{z}(t))$ with eigenvalue $\overline{\lambda}_s(\mathbf{z}(t))$, we have that

$$\overline{\mathbf{u}}_s(1) = \overline{\gamma}_s(C)\overline{\mathbf{u}}_s(0), \tag{5.9}$$

where $\overline{\gamma}_s(C) = \pm 1$. γ_r and $\overline{\gamma}_s$ may be regarded as the holonomies of the real eigenvector line bundles E_r and $\overline{E}_s$ over R,

$$\begin{aligned} E_r &= \{(\mathbf{z}, \mathbf{u}) \in R \times \mathbb{R}^n |\, (\mathrm{L}(\mathbf{z}) - \lambda_r(\mathbf{z})) \cdot \mathbf{u} = 0\}, \\ \overline{E}_s &= \{(\mathbf{z}, \mathbf{u}) \in R \times \mathbb{R}^n |\, (\overline{\mathrm{L}}(\mathbf{z}) - \overline{\lambda}_s(\mathbf{z})) \cdot \mathbf{u} = 0\}. \end{aligned} \tag{5.10}$$

They are examples of (real) geometric phases [8], though in this context (holonomies of eigenvectors of real symmetric matrices) have a long history (see, eg, [5] and [9]).

For the periodic Toda chain, the Maslov index and eigenvector holonomies are related by the following:

Theorem 5.1.

$$(-1)^{\mu/2} = \prod_{r\ even} \gamma_r \prod_{s\ even} \overline{\gamma}_s. \tag{5.11}$$

We note that the unrestricted product $\prod_r \gamma_r$ is always $+1$, since this gives the holonomy of the (trivial) determinant bundle of $R \times \mathbb{R}^n$. Similarly for the unrestricted product $\prod_s \overline{\gamma}_s$. Thus, either of the products in (5.11) may be restricted to odd rather than even indices. (Alternatively, from Proposition 4.2 one can deduce that $\gamma_r = \gamma_{r+1}$ for r even and $\overline{\gamma}_r = \overline{\gamma}_{r+1}$ for r odd.)

Proof. Let C be a continuous closed curve in the regular set R of F, and let S be a transverse disk with boundary C. Let N denote the number of singular points in the image of S. From (5.7),

$$(-1)^{\mu(C)/2} = (-1)^N. \tag{5.12}$$

At the singular points $\mathbf{z}_j \in \Sigma_1$, there is precisely one doubly degenerate eigenvalue of either $\mathrm{L}(\mathbf{z}_j)$ or $\overline{\mathrm{L}}(\mathbf{z}_j)$. For definiteness, suppose λ_r and λ_{r+1} are degenerate at $\mathbf{z}_*$. As in (4.10) and (4.11), in a neighbourhood of $\mathbf{z}_*$, $\overline{\mathrm{L}}(\mathbf{z})$ is smoothly conjugate to a diagonal matrix, while $\mathrm{L}(\mathbf{z})$ is smoothly conjugate to a block diagonal matrix $\Lambda(\mathbf{z})$ with a single two-dimensional block, with elements $\Lambda_{ij}(\mathbf{z})$, $i, j = 1, 2$, and the rest diagonal. Let

$$\xi(\mathbf{z}) = \tfrac{1}{2}(\Lambda_{11}(\mathbf{z}) - \Lambda_{22}(\mathbf{z})), \quad \eta(\mathbf{z}) = \Lambda_{12}(\mathbf{z}). \tag{5.13}$$

As in Proposition 4.4, ξ and η are functionally independent near $\mathbf{z}_*$, and $\xi(\mathbf{z}_*) = \eta(\mathbf{z}_*) = 0$. Construct local coordinates $(\xi, \eta, a_1, \ldots, a_{2n-2})$ with origin at $\mathbf{z}_*$. Let C_* denote the closed curve parameterised by $\xi(t) = \epsilon \cos 2\pi t$, $\eta(t) = \epsilon \sin 2\pi t$, $a_k = 0$, with ϵ chosen small enough so that C_* lies in the coordinate neighbourhood. Along C_*, let $\Lambda^{(2)}(t)$ denote the two-dimensional block of Λ. Then

$$\Lambda^{(2)}(t) = \tfrac{1}{2}\tau(t)\mathrm{I} + \begin{pmatrix} -\cos 2\pi t & \sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix}, \tag{5.14}$$

where $\tau(t)$ is given by $\Lambda_{22} + \Lambda_{11}$ along C_*. The eigenvectors $(\cos \pi t \ \ \sin \pi t)^T$ and $(\sin \pi t \ \ -\cos \pi t)^T$ of $\Lambda^{(2)}(t)$ change sign around C_*. This corresponds to holonomies

$$\gamma_r(C_*) = \gamma_{r+1}(C_*) = -1 \tag{5.15}$$

in the associated eigenvectors $\mathbf{u}_r$ and $\mathbf{u}_{r+1}$ of L. The other holonomies, ie $\gamma_{r \neq p}(C_*)$, $\overline{\gamma}_s(C_*)$, are trivially $+1$.

We introduce contours C_j analogously for all singularities $\mathbf{z}_j$ in the image of S (replacing L by $\overline{\mathrm{L}}$ in the preceding as appropriate). C is homologous to an oriented sum of circuits C_j. Around each C_j, exactly one even-indexed eigenvector holonomy of either L or $\overline{\mathrm{L}}$ is -1 (cf (5.15)), and the rest are $+1$. Thus,

$$\prod_{r,even} \gamma_r(C) \prod_{s,even} \gamma_s(C) = (-1)^N. \tag{5.16}$$

Comparing (5.16) and (5.12), we obtain the result (5.11). $\square$

6 Discussion

Σ_k, the space of codimension-$2k$ singularities of the Toda chain, is a symplectic submanifold corresponding to points where there are k doubly degenerate eigenvalues of the Lax matrices L and $\overline{\mathrm{L}}$, for $1 \leq k \leq n-1$. These submanifolds are of elliptic type, and we have calculated the frequencies of transverse oscillations under integrable flows that preserve them pointwise. The codimension-two singularities are sources for the Maslov index. The (even) Maslov index of a closed curve C is determined, modulo 4, by the product of the even- (or odd-) indexed eigenvector holonomies of L and $\overline{\mathrm{L}}$. It would be interesting to relate higher Maslov classes [23, 25] to singularities of higher codimension, and to compute these higher Maslov classes explicitly for the Toda chain. In this context, it would also be interesting to study higher-order corrections [10, 11] to the semiclassical quantization conditions (5.2) for the quantum Toda chain [17, 18].

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References

[1] M. Adler and P. van Moerbeke. Completely integrable systems, Euclidean lie algebras and curves. *Adv. Math.*, 38:267–317, 1980.

[2] M. Adler and P. van Moerbeke. Linarization of Hamiltonian systems, Jacobi varieties and representation theory. *Adv. Math.*, 38:318–379, 1980.

[3] M. Adler and P. van Moerbeke. The Toda lattice, Dynkin diagrams, singularities and Abelian varieties. *Adv. Math.*, 38:318–379, 1980.

[4] V.I. Arnold. Characteristic class entering quantization conditions. *Functional Anal. Appl.*, 1:1–13, 1967.

[5] V.I. Arnold. *Mathematical methods of classical mechanics.* Springer-Verlag, 2nd edition, 1989.

[6] M. Audin. *Spinning Tops.* Cambridge University Press, 1996.

[7] O. Babelon, D. Bernard, and M. Talon. *Introduction to classical integable systems.* Cambidge University Press, 2003.

[8] M.V. Berry. Quantal phase factors accompanying adiabatic changes. *Proc. Roy. Soc. London Ser. A*, 392:45–57, 1984.

[9] M.V. Berry. Anticipations of the geometric phase. *Physics Today*, 43:34–40, 1990.

[10] M. Cargo, A. Gracia-Saz, R.G. Littlejohn, M.W. Reinsch, and P. de M. Rios. Quantum normal forms, Moyal star product and Bohr-Sommerfeld approximation. *J. Phys. A*, 38:1977–2004, 2005.

[11] Y. Colin de Verdiére. Bohr-Sommerfeld rules to all orders. to be published in *Ann. Inst. Henri Poincaré*, year = 2004.

[12] L.H. Eliasson. Normal form for Hamiltonian systems with Poisson commuting integrals – elliptic case. *Comm. Math. Helv.*, 65:4–35, 1990.

[13] H. Flaschka. The toda lattice. i. existence of integrals. *Phys. Rev. B*, 9:1924–1925, 1974.

[14] H. Flaschka. Integrable systems and torus actions. In P. Cartier O. Babelon and Y. Kosmann-Schwarzbach, editors, *Lectures on integrable systems*. World Scientific, 1994.

[15] A. Fomenko. *Topological classification of integrable systems*. Number 6 in Advances in Soviet Mathematics. AMS, 1991.

[16] J.A. Foxman and J.M. Robbins. The Maslov index and nondegenerate singularities of integrable systems. preprint, 2004.

[17] M.C. Gutwiller. The quantum mechanical Toda lattice. *Ann. Physics*, 124, 1980.

[18] M.C. Gutwiller. The quantum mechanical Toda lattice. ii. *Ann. Physics*, 133:304–331, 1981.

[19] M. Hénon. Integrals of the Toda lattice. *Phys. Rev. B*, 9:1921–1923, 1974.

[20] J.B. Keller. Corrected Bohr-Sommerfeld quantum conditions for non-separable systems. *Ann. Phys.*, 4:180–188, 1958.

[21] V.P. Maslov and M. Fedoriuk. *Semiclassical approximation in quantum mechanics*. Reidel, 1981.

[22] L.S. Polyakova. Topological invariants for some algebraic analogs of the toda lattice. In *Topological classification of integrable systems*, volume 6 of *Adv. Sov. Math.*, pages 185–207. 1991.

[23] H. Suzuki. Residue classes of Lagrangian subbundles and Maslov classes. *Trans. Amer. Math. Soc.*, 347:189–202, 1995.

[24] M. Toda. Waves in a nonlinear lattice. *Prog. Theor. Phys. Suppl.*, 45:174–200, 1970.

[25] V.V. Trofimov. Generalized Maslov classes on the path space of a symplectic manifold. *Proc. Steklov Institute of Mathematics*, 205:157–179, 1995.

[26] P. van Moerbeke and D. Mumford. The spectrum of difference operators and algebraic curves. *Acta Math.*, 143:93–154, 1979.

[27] J. Vey. Sur certains systèmes dynamiques séparables. *Amer. J. Math.*, 100:591–614, 1978.

[28] Nguyen Tien Zung. Symplectic topology of integrable Hamiltonian systems, I: Arnold-Liouville with singularities. *Compositio Math.*, 101:179–215, 1996.